Phys 410
Fall 2015
Lecture #23 Summary
17 November, 2015

We began a discussion of coupled oscillators by considering two masses on a friction-less surface, with 3 springs between them, all in a line. The left mass (m_1) is connected to the left wall by a spring of spring constant k_1, while the other mass (m_2) is connected to the right wall by a spring of spring constant k_3. The two masses are also directly connected to each other by a third spring characterized by k_2. In the absence of spring 2, the two masses would oscillate independently at their own natural frequencies. However, with the coupling, they will have a new type of motion characterized by 'normal modes.'

We wrote down the Lagrangian of the system and found that Lagrange's equations yield a pair of coupled second-order linear differential equations: $-(k_1 + k_2)x_1 + k_2x_2 = m_1\ddot{x}_1$, and $k_2x_1 - (k_2 + k_3)x_2 = m_2\ddot{x}_2$. These equations can be summarized in an elegant 2x2 matrix equation: $\bar{\bar{M}}\ddot{\vec{x}} = -\bar{\bar{K}}\vec{x}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of unknowns, $\bar{\bar{M}} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ is the "mass matrix", and $\bar{\bar{K}} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}$ is the "spring constant matrix". This equation is a generalization of the mass on a spring equation. In fact it reduces to two un-coupled mass/spring equations when $k_2 = 0$.

Although the two un-coupled masses would oscillate on their own at different frequencies, we are going to try an *ansatz* in which both masses oscillate together at a single frequency. We use the complex form, which worked so well for the single harmonic oscillator, but now generalized to 2 oscillators: $\vec{x}(t) = Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, and C_1 and C_2 are complex constants. Putting this into the matrix equation yields $\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right)\vec{C} = 0$. This is similar to, but not exactly, an eigenvalue problem (the two different values of mass prevents it from being an eigenvalue problem). Nevertheless, we can still use the formalism of linear algebra to solve this problem. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right) = 0$. This yields a quadratic equation for ω^2, with two solutions.

We then specialized to the case of equal masses (m) and equal spring constants (k). The quadratic equation then yields two normal mode frequencies: $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. The corresponding normal modes were found to be $x_1 = x_2 = A\cos(\omega_1 t - \delta_1)$ for ω_1 (this is the 'sloshing mode') and $x_1 = -x_2 = A\cos(\omega_2 t - \delta_2)$ for ω_2 (this is the 'beating mode'). The general solution is a linear combination of these two normal modes with arbitrary weighting constants and phases.

The choice of two new coordinates, so-called normal coordinates, would have diagonalized the $\overline{\overline{K}}$ matrix from the get-go. In this case the normal coordinates are $\xi_1 = \frac{1}{2}(x_1 + x_2)$, and $\xi_2 = \frac{1}{2}(x_1 - x_2)$. Each obeys an un-coupled equation of motion and the two 'oscillators' have normal mode frequencies of $\omega_1 = \sqrt{k/m}$, and $\omega_2 = \sqrt{3k/m}$. More generally, a transformation of coordinates that simultaneously diagonalizes the mass and spring constant matrices will reveal the normal coordinates.

We then considered another coupled oscillator problem – the double pendulum. We wrote down the Lagrangian, which turned out to be quite complicated. It leads to nonlinear equations of motion – as is well known for the single pendulum. To avoid this problem (which we will deal with later in the discussion of nonlinear dynamics), we made a "small oscillations" approximation for the double pendulum. In this approximation we take ϕ_1, ϕ_2, $\dot{\phi}_1$, and $\dot{\phi}_2$ to be small, and only keep terms up to second order in these quantities. We then did a Taylor series expansion for the kinetic energy and potential energy to arrive at an approximate Lagrangian of the form: $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\left(L_1\dot{\phi}_1\right)^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} m_2 \left(L_2 \dot{\phi}_2\right)^2 - \frac{(m_1+m_2) g L_1 \phi_1^2}{2} - \frac{m_2 g L_2 \phi_2^2}{2}$. Note that both the kinetic energy and the potential energy are homogeneous quadratic functions.

We then used Lagrange's equations to find the equations of motion for the two generalized coordinates ϕ_1, ϕ_2, with the following results:

ϕ_1-equation: $-(m_1 + m_2) g L_1 \phi_1 = (m_1 + m_2) L_1^2 \ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2$

ϕ_2-equation: $-m_2 g L_2 \phi_2 = m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2$

These two equations can be summarized in matrix form as $\overline{\overline{M}}\ddot{\vec{\phi}} = -\overline{\overline{K}}\vec{\phi}$, with $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, $\overline{\overline{M}} = \begin{pmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}$ and $\overline{\overline{K}} = \begin{pmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}$. The "mass matrix" is now made up of rotational inertia terms, while the "spring constant matrix" is made up of restoring torque terms. Note that the K-matrix is diagonal, whereas the M-matrix is not – this is the opposite of the situation for the 2-mass-3-spring problem, showing that we have a different kind of coupling here. We again use the complex *ansatz* for the solution vector: $\vec{\phi}(t) = Re\left[\vec{C} e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, and C_1 and C_2 are complex constants. Putting this into the matrix equation yields $\left(\overline{\overline{K}} - \omega^2 \overline{\overline{M}}\right)\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\overline{\overline{K}} - \omega^2 \overline{\overline{M}}\right) = 0$. This yields a quadratic equation for ω^2, with two solutions.

We then considered the special case of a double pendulum with equal masses (m) and equal lengths (L), and introduce the natural frequency ($\omega_0^2 \equiv g/L$). The determinant yields two normal mode frequency solutions: $\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$, and $\omega_2 = \omega_0\sqrt{2+\sqrt{2}}$. The corresponding normal modes are the analogs of the "sloshing" and "beating" modes. The first is of the form $\vec{\phi} = A_1\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\cos(\omega_1 t - \delta_1)$, while the second is $\vec{\phi} = A_2\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\cos(\omega_2 t - \delta_2)$. In the first normal mode the two pendula swing together in phase (the sloshing mode), with the lower pendulum swinging with greater amplitude. In the other mode the two pendula swing 180° out of phase (a type of beating mode).